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A new domain decomposition method for an HJB equation☆

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Abstract

In this paper we propose a new domain decompostion method for solving a Hamilton–Jacobi–Bellman equation of second order. The basic idea is to solve an equivalent quasivariational inequality instead of the original discretized HJB equation.

Keywords: HJB equation; Domain decomposition method; Quasivariational inequality; Convergence

1. Introduction

Consider the following Hamilton–Jacobi–Bellman (HJB) equation:

$$\begin{cases} \max\limits_{v\in V}\left\{-\sum\limits_{i,j=1}^{2} a_{ij}(x,v)\partial_{ij}u+\sum\limits_{i=1}^{2} g_i(x,v)\partial_i u+\alpha u-f(x,v)\right\}=0 & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$, matrix (a_{ij}) is symmetric and nonnegative definite, $a_{11}, a_{22} > 0$, constant $\alpha \geqslant 0$. Furthermore, we assume [7]

$$a_{11}, a_{22} \geqslant |a_{12}|. \tag{1.2}$$

Eq. (1.1) is a full nonlinear equation arising in solving optimal control problems by dynamic programming technique, see [1,4].

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In order to solve (1.1) numerically we need two discretizations:

(i) discretize control parameter $v : v_1, \dots, v_k$;
(ii) discretize PD operator, usually, by finite-difference method.

Then we obtain the discrete problem of (1.1) as follows: find $u \in \mathbb{R}^n$ such that

$$\max_{1 \leqslant j \leqslant k} \{A^j u - f^j\} = 0, \tag{1.3}$$

where $A^j \in \mathbb{R}^{n\times n}$, $f^j \in \mathbb{R}^n$, $j = 1, \dots, k$. Eq. (1.3) is a system of nonsmooth nonlinear equations. If we use an appropriate upwind finite-difference scheme (see [7] and the references therein) then for the case $\alpha > 0$ [7] has proved that (1.3) satisfies the following condition.

Condition A: $A^j = (a^j_{ls})$, $j = 1, \dots, k$, are L-matrices (i.e., $a^j_{ls} \leqslant 0$ for $s \neq l$ and $a^j_{ll} > 0$, $l = 1, \dots, n$), and have strict diagonal dominance (see [8]).

For the case $\alpha = 0$ we propose the following condition.

Condition B: A^j, $j = 1, \dots, k$, are L-matrices, have weak diagonal dominance [8], satisfy

$$a^j_{11} > \sum_{s=2}^{n} |a^j_{1s}|, \quad j = 1, \dots, k \tag{1.4}$$

and all the matrices $A(p_1, \dots, p_n) = (a^{p_l}_{ls})$, $p_l = 1, \dots, k$, $l = 1, \dots, n$ satisfy

$$A(p_1, \dots, p_n) \quad \text{are irreducible.} \tag{1.5}$$

Remark. If we number the mesh points in such a way that the mesh point with No. 1 is a neighbor of a boundary mesh point then (1.4) holds. On the other hand, since we discretize the PD operators in (1.1) by same difference scheme, A^j, $j = 1, \dots, k$, have same nonzero element distribution and (1.5) holds. Hence condition B is satisfied in our case.

Domain decomposition method (DDM) is one of the most important algorithm for solving discrete problems of PDEs. One of its advantages is that it is able to be easily parallelized and has good parallel performance (see [6], for example). Camilli et al. [2] has proposed a DDM for solving the discrete problems of a kind of HJB equations of the first order. Sun [7] has given a DDM for solving (1.3) (the discrete problem of (1.1)) directly under condition A (the case $\alpha > 0$).

In this paper we propose a new DDM for solving (1.3) under condition A or B ($\alpha > 0$ or $\alpha = 0$). Instead of (1.3), we solve a quasivariational inequality (QVI) equivalent to (1.3). Furthermore, we have proved the existence and uniqueness of solution for (1.3) in the two cases.

2. Equivalent QVI

In order to establish the QVI equivalent to (1.3) we define operator B as follows:

$$Bv = \min_{2 \leqslant j \leqslant k} \{(I - \tau^j A^j)v + \tau^j f^j\}, \quad \forall v \in \mathbb{R}^n,$$

where I is the identity matrix, $\tau^j \in (0, \infty)$, $j = 2, \dots, k$. Consider the following quasivariational inequality:

$$u \leqslant Bu, \quad (A^1 u - f^1, v - u) \geqslant 0, \quad \forall v \leqslant Bu. \tag{2.1}$$

Theorem 2.1. *HJB equation (1.3) is equivalent to quasivariational inequality (2.1).*

Proof. Let $(v)_i$ be the ith component of $v \in \mathbb{R}^n$. Obviously, (1.3) is equivalent to

$$A^j u - f^j \leqslant 0, \quad j = 1, \dots, k,$$

$$\prod_{j=1}^{k} (A^j u - f^j)_i = 0, \quad i = 1, \dots, n,$$

which is equivalent to

$$A^1 u - f^1 \leqslant 0, \tag{2.2}$$

$$u \leqslant (1 - \tau^j A^j) u + \tau^j f^j, \quad j = 2, \dots, k, \tag{2.3}$$

$$(A^1 u - f^1)_i \prod_{j=2}^{k} (u - (I - \tau^j A^j) u - \tau^j f^j)_i = 0, \quad i = 1, \dots, n. \tag{2.4}$$

Eq. (2.3) is equivalent to $u \leqslant Bu$. But under the condition $u \leqslant Bu$ we know that $u - Bu = 0$ is equivalent to

$$\prod_{j=2}^{k} (u - (I - \tau^j A^j) u - \tau^j f^j)_i = 0, \quad i = 1, \dots, n.$$

Therefore (2.2)–(2.4) is equivalent to

$$A^1 u - f^1 \leqslant 0, \quad u \leqslant Bu, \quad (A^1 u - f^1)^T (u - Bu) = 0,$$

which is just (2.1). The proof is complete. □

Now we study a property of operator B. An operator C is called order-preserved if $w, v \in \mathbb{R}^n$ and $w \leqslant v$ implies $Cw \leqslant Cv$.

Theorem 2.2. *If*

$$\tau^j \leqslant \left(\max_{1 \leqslant l,s \leqslant n} |a_{ls}^j| \right)^{-1}, \quad j = 2, \dots, k \tag{2.5}$$

then the operator B is continuous and order-preserved.

Proof. It is trivial that B is continuous. Obviously, $I - \tau^j A^j$, $j = 1, \dots, k$, are nonnegative if (2.5) holds. Then

$$(I - \tau^j A^j) w \leqslant (I - \tau^j A^j) v \quad \text{if } w \leqslant v, \; j = 2, \dots, k,$$

which implies

$$\min_{2 \leqslant j \leqslant k} \{(I - \tau^j A^j) w + \tau^j f^j\} \leqslant \min_{2 \leqslant j \leqslant k} \{(I - \tau^j A^j) v + \tau^j f^j\}.$$

It is just $Bw \leqslant Bv$. The proof is complete. □

Corollary. *If A^j, $j=2,\dots,k$ are L-matrices and*

$$\tau^j \leqslant \left(\max_{1\leqslant s\leqslant n} a_{ss}^j\right)^{-1}, \quad j=2,\dots,k, \tag{2.6}$$

then B is order-preserved.

3. Algorithm

For simplicity we discuss only the case of two subdomains. Discretize (1.1) as in Section 1, and obtain (1.3). Decompose $\Omega=\Omega_1\cup\Omega_2$ such that $\Omega_1\cap\Omega_2=\phi$. Denote $N=\{1,\dots,n\}$, $N=N_1\cup N_2$, where N_i is the index set corresponding to the mesh points in Ω_i, $i=1,2$.

Now we introduce the concept of subsolution for (1.3). We call $v\in\mathbb{R}^n$ a subsolution for (1.3) if

$$\max_{1\leqslant j\leqslant k}\{A^j v - f^j\} \leqslant 0. \tag{3.1}$$

We denote by S the set of all the subsolution for (1.3).

Lemma 3.1. *Assume condition A is satisfied. Let $v=\theta e$, $e=(1,\dots,1)^T$. If*

$$\theta \leqslant \min_{\substack{1\leqslant j\leqslant k\\ 1\leqslant l\leqslant n}} \left(a_{ll}^j + \sum_{s\neq l} a_{ls}^j\right)^{-1} f_l^j \tag{3.2}$$

then v is a subsolution for (1.3).

Proof. Condition A implies

$$a_{ll}^j + \sum_{s\neq l} a_{ls}^j > 0, \quad j=1,\dots,k;\ \ l=1,\dots,n. \tag{3.3}$$

Eq. (3.3) combining with (3.2) yields (3.1) easily.

Algorithm DDM

Step 1: Given $\varepsilon>0$, $u^0\in S$, $m:=0$;

Step 2: Solve parallelly the following subproblems:

$$u^{m,i}\in K_i^m,$$
$$(A^1u^{m,i}-f^1, v-u^{m,i})\geqslant 0, \quad \forall v\in K_i^m, \tag{3.4}$$

where $K_i^m=\{v\in\mathbb{R}^n : v_s=u_s^m \text{ if } s\in N\backslash N_i,\ v_s\leqslant (Bu^m)_s \text{ if } s\in N_i\}$, $i=1,2$;

Step 3: $u^{m+1}=\max\{u^{m,1},u^{m,2}\}$;

Step 4: If $\|u^{m+1}-u^m\|\leqslant\varepsilon$ then stop otherwise $m:=m+1$, go to step 2.

Remark. First, we can easily choose $u^0\in S$ by Lemma 3.1. Second, (3.4) is not a QVI but an ordinary variational inequality which has a unique solution [3] and can be solved by many well-known algorithms (see [5,9] and the references therein) under condition A or B.

4. Convergence: condition A case

At first we introduce a lemma [8].

Lemma 4.1. *If A is an L-matrix and has strong diagonal dominance then A is an M-matrix.*

Now we prove the uniqueness of solution for (1.3).

Theorem 4.1. *Assume condition A is satisfied. Then (1.3) has at most one solution.*

Proof. Assume (1.3) has solution u and $\tilde{u}$. Then there exist indexes $p_1,\dots,p_n$, $1 \leqslant p_l \leqslant k$, $l=1,\dots,n$ such that

$$\sum_{s=1}^{n} a_{ls}^{p_l} u_s - f_l^{p_l} = 0, \quad l=1,\dots,n$$

or the matrix form

$$A^* u - f^* = 0,$$

where $A^* = A(p_1,\dots,p_n) = (a_{ls}^{p_l})$, $f^* = (f_l^{p_l})$. It is easy to see that A^* is also an L-matrix and has strong diagonal dominance. So A^* is an M-matrix by Lemma 4.1. On the other hand, we have

$$A^* \tilde{u} - f^* \leqslant 0.$$

Hence,

$$A^* u \geqslant A^* \tilde{u},$$

which combining with that A^* is an M-matrix implies that $u \geqslant \tilde{u}$. Similarly we obtain $u \leqslant \tilde{u}$. Therefore, $u = \tilde{u}$ and the proof is complete. □

In order to prove the convergence we need one more lemma as follows.

Lemma 4.2. *Assume C is an M-matrix, $N = I \cup J$, $I \cap J = \phi$, $K_i = \{v \in \mathbb{R}^n : v \leqslant \psi^i,\ v_s = \psi_s^i \text{ if } s \in J\}$. If $\psi^1 \geqslant \psi^2$ and*

$$u^i \in K_i,$$

$$(Cu^i - f, v - u^i) \geqslant 0, \quad \forall v \in K_i,$$

$i = 1,2$, *then* $u^1 \geqslant u^2$.

Proof. Let $I_1 = \{s \in I : u_s^1 = \psi_s^1\}$. Then $u_s^1 = \psi_s^1 \geqslant \psi_s^2 \geqslant u_s^2$ if $s \in I_1$. On the other hand,

$$(Cu^1 - f)_s = 0, \quad (Cu^2 - f)_s \leqslant 0 \quad \text{if } s \in I \backslash I_1. \tag{4.1}$$

Denote by f_P the subvector of f corresponding to the index set P, and by $C_{P,Q}$ the submatrix of C corresponding to the row and column index set P and Q. Then (4.1) implies that

$$C_{I\backslash I_1,N}(u^1 - u^2) \geqslant 0.$$

But $C_{I\backslash I_1,N} = C_{I\backslash I_1,I\backslash I_1} + C_{I\backslash I_1,J\cup I_1}$ and $C_{I\backslash I_1,J\cup I_1} \leqslant 0$ (M-matrix must be L-matrix [8]). Hence,

$$C_{I\backslash I_1,I\backslash I_1}(u^1 - u^2)_{I\backslash I_1} \geqslant - C_{I\backslash J_1,J\cup I_1}(u^1 - u^2)_{J\cup I_1} \geqslant 0,$$

which means $(u^1 - u^2)_{I\backslash I_1} \geqslant 0$ because $C_{I\backslash I_1,I\backslash I_1}$ is an M-matrix also. Finally we have $u^1 \geqslant u^2$. □

Theorem 4.2. *Assume condition A and (2.6) are satisfied, u^m is generated by Algorithm DDM. Then $\{u^m\}$ monotonically increasing, convergent to the unique solution of (1.3).*

Proof. At first we compare u^0 with $u^{0,1}$. If $s \in N\backslash N_1$ then $u^{0,1}_s = u^0_s$ by (3.4). Let $J_{11} = \{s \in N_1 : u^{0,1}_s = (Bu^0)_s\}$. Then for any $s \in J_{11}$ there exist an integer q, $2 \leqslant q \leqslant k$, such that

$$\begin{aligned} u^{0,1}_s &= (Bu^0)_s = [(I - \tau^q A^q)u^0 + \tau^q f^q]_s \\ &= u^0_s - \tau^q(A^q u^0 - f^q)_s \geqslant u^0_s. \end{aligned}$$

For any $s \in N_1\backslash J_{11}$ we have $u^{0,1}_s < (Bu^0)_s$. Hence, taking v in (3.4) such that $v_s = u^m_s$ if $s \in N\backslash N_1, v_s = (Bu^0)_s$ if $s \in J_{11}$, and $v_s = u^{0,1}_s \pm \varepsilon_s$ if $s \in N_1\backslash J_{11}$, where $0 \leqslant \varepsilon_s < (Bu^0)_s$ and only one $\varepsilon_s > 0$ every time, we may derive that

$$(A^1 u^{0,1} - f^1)_{N_1\backslash J_{11},N} = 0. \tag{4.2}$$

On the other hand, since $u^0 \in S$ we have

$$(A^1 u^0 - f^1)_{N_1\backslash J_{11},N} \leqslant 0,$$

which combining with (4.2) yields

$$[A^1(u^{0,1} - u^0)]_{N_1\backslash J_{11},N} \geqslant 0. \tag{4.3}$$

Noting $u^{0,1}_s = u^0_s$ if $s \in N\backslash N_1, u^{0,1}_s \geqslant u^0_s$ if $s \in J_{11}$, and $a^1_{ls} \leqslant 0$ if $l \in N_1\backslash J_{11}$, we obtain

$$[A^1(u^{0,1} - u^0)]_{N_1\backslash J_{11},N\backslash N_1} = 0, \quad [A^1(u^{0,1} - u^0)]_{N_1\backslash J_{11},J_{11}} \leqslant 0. \tag{4.4}$$

It follows from (4.3) and (4.4) that

$$[A^1(u^{0,1} - u^0)]_{N_1\backslash J_{11},N_1\backslash J_{11}} \geqslant 0,$$

which implies $u^{0,1}_s \geqslant u^0_s$ for $s \in N_1\backslash J_{11}$ since $A^1_{N_1\backslash J_{11},N_1\backslash J_{11}}$ is an M-matrix also. Therefore, we have proved $u^{0,1} \geqslant u^0$. Similarly we obtain $u^{0,2} \geqslant u^0$. Hence

$$u^1 \geqslant u^0, \quad Bu^1 \geqslant Bu^0, \tag{4.5}$$

by the definition of u^1 and the corollary of Theorem 2.2. □

Now it follows from Lemma 4.2, (3.4) and (4.5) that $u^{1,i} \geqslant u^{0,i}$. Hence $u^2 \geqslant u^1$ and $Bu^2 \geqslant Bu^1$. Then we derive by induction that

$$u^{m+1} \geqslant u^{m,i} \geqslant u^m \geqslant u^{m-1,i}, \qquad Bu^{m+1} \geqslant Bu^m, \qquad m = 1, 2, \ldots\ . \tag{4.6}$$

Similarly to the proof of Lemma 3.1 we can prove that there exists $b \in \mathbb{R}^n$ such that

$$A^j b - f^j \geqslant 0, \quad j = 1, \ldots, k.$$

Then similarly to the argument for $u^{0,i} \geqslant u^0$ above we obtain $u^{0,i} \leqslant b$, $i = 1,2$, and $u^1 \leqslant b$. By induction we derive that

$$u^{m,i}, u^m \leqslant b, \quad m = 0,1,\ldots, \tag{4.7}$$

which and (4.6) implies that there exists $u^* \in \mathbb{R}^n$ such that $u^{m,i}, u^m \to u^*(m \to \infty)$. Let $m \to \infty$ in (3.4) we obtain that $u^* \in K_i^*$ and

$$(A^1u^* - f^1, v - u^*) \geqslant 0, \quad \forall v \in K_i^*, \tag{4.8}$$

where $K_i^* = \{v \in \mathbb{R}^n : v_s = u_s^* \text{ if } s \in N \backslash N_i, v_s \leqslant (Bu^*)_s \text{ if } s \in N_i.\}$, $i = 1,2$.

At last we prove u^* is the solution of (1.3), which is unique by Theorem 4.1. For any $\tilde{v} \leqslant Bu^*$ we have

$$\begin{aligned}(A^1u^* - f^1, \tilde{v} - u^*) &= \sum_{s=1}^{n} (A^1u^* - f^1)_s(\tilde{v}_s - u_s^*) \\ &= \sum_{s\in N_1} + \sum_{s \in N\backslash N_1} .\end{aligned} \tag{4.9}$$

Taking v in (4.8) for $i = 1$ such that $v_s = u_s^*$ if $s \in N\backslash N_1$ and $v_s = \tilde{v}_s$ if $s \in N_1$ we obtain

$$\sum_{s \in N_1} (A^1u^* - f^1)_s(\tilde{v}_s - u_s^*) \geqslant 0. \tag{4.10}$$

Taking v in (4.8) for $i = 2$ such that $v_s = u_s^*$ if $s \in N_1$ and $v_s = \tilde{v}_s$ if $s \in N\backslash N_1$ we obtain

$$\sum_{s \in N\backslash N_1} (A^1u^* - f^1)_s(\tilde{v}_s - u_s^*) \geqslant 0. \tag{4.11}$$

It follows from (4.9)–(4.11) that

$$(A^1u^* - f^1, \tilde{v} - u^*) \geqslant 0, \quad \forall \tilde{v} \leqslant Bu^*,$$

which means u^* is the unique solution of (1.3) by Theorem 2.1. The proof is complete. □

Remark. We have proved also the existence of the solution of (1.3) in the proof above.

5. Convergence: condition B case

At first we introduce a lemma similar to Lemma 4.1.

Lemma 5.1 (Young [8]). *If A is an irreducible L-matrix and has weak diagonal dominance then A is an M-matrix.*

Then by the argument similar to that for Theorem 4.1 we can prove

Theorem 5.1. *Assume condition B is satisfied. Then* (1.3) *has at most one solution.*

Now we may prove directly the existence of the solution for (1.3).

Theorem 5.2. *Assume condition B is satisfied. Then* (1.3) *has a unique solution.*

Proof. Split A^j into $A^j = D^j - C^j$, where $D^j = \operatorname{diag} A^j$. Let $A_t^j = D^j - tC^j$, $j = 1,\dots,k$, $0 \leqslant t \leqslant 1$. Then $D^j \geqslant 0$, $C^j \geqslant 0$, and A_t^j, $j = 1,\dots,k$, have strong diagonal dominance for $0 \leqslant t < 1$. Denote

$$F_t u = \max_{1 \leqslant j \leqslant k} \{A_t^j u - f^j\}, \quad 0 \leqslant t \leqslant 1.$$

Then $F_1 u = 0$ is just (1.3), and for every $0 \leqslant t < 1$ equation $F_t u = 0$ has an unique solution u_t by Theorem 4.2. Similarly to the proof of Theorem 4.1 we derive that for every $t \in [0,1)$ there exist $A_t^* = A_t(p_1,\dots,p_n) = (a_{tls}^{p_l})$ and $f_t^* = (f_l^{p_l})$ such that

$$A_t^* u_t = f_t^*, \tag{5.1}$$

where $p_1,\dots,p_n$ depend of t. For every fixed $t \in [0,1)$ we have

$$A_1(p_1,\dots,p_n) = (a_{ls}^{p_s}) \leqslant A_t(p_1,\dots,p_n). \tag{5.2}$$

It is known that $A_t(p_1,\dots,p_n)$, $0 \leqslant t < 1$, are M-matrices, and $A_t(p_1,\dots,p_n)^{-1} \geqslant 0$ $(0 \leqslant t < 1)$. On the other hand, it follows from condition B and Lemma 5.1 that $A_1(p_1,\dots,p_n)$ is an M-matrix, and $A_1(p_1,\dots,p_n)^{-1} \geqslant 0$. Therefore, we may multiply (5.2) right by $A_t(p_1,\dots,p_n)^{-1}$, left by $A_1(p_1,\dots,p_n)^{-1}$, and obtain

$$0 \leqslant A_t(p_1,\dots,p_n)^{-1} \leqslant A_1(p_1,\dots,p_n)^{-1}. \tag{5.3}$$

From (5.1) and (5.3) we derive for $t \in [0,1)$ that

$$\begin{aligned}
\|u_t\|_1 &\leqslant \|A_t(p_1,\dots,p_n)^{-1}\|_1 \cdot \|f_t^*\|_1 \\
&\leqslant \|A_1(p_1,\dots,p_n)^{-1}\|_1 \cdot \|f_t^*\|_1 \\
&\leqslant \max_{\substack{1 \leqslant p_l \leqslant k \\ 1 \leqslant l \leqslant n}} \|A_1(p_1,\dots,p_n)^{-1}\|_1 \cdot \|f_t^*\|_1,
\end{aligned}$$

which means $\{u_t : 0 \leqslant t < 1\}$ is bounded in $\mathbb{R}^n$.

Hence there exist $t_m \in [0,1)$, $m = 1,2,\dots,$ and $u^* \in \mathbb{R}^n$ such that $t_m \to 1$ $(m \to \infty)$,

$$F_{t_m} u_{t_m} = 0 \tag{5.4}$$

and $u_{t_m} \to u^*(m \to \infty)$. Obviously, F_t is continuous respect to t. Let $m \to \infty$ in (5.4) and conclude that $F_1 u^* = 0$, i.e., u^* is the solution of (1.3). The proof is complete. □

In order to prove the convergence in this case we need one more lemma.

Lemma 5.2. *Assume condition B and* (2.6) *are satisfied*, u^m *is generated by Algorithm DDM. Then* $u^m \leqslant u$.

Proof. At first we prove that $u^0 \leqslant u$. Using the same argument as that in Theorem 4.1 we know

$$A^* u - f^* = 0.$$

It follows from the definition of subsolution that

$$A^*u^0 - f^* \leqslant 0.$$

Hence $A^*(u^0 - u) \leqslant 0$, and $u^0 \leqslant u^*$ since A^* is an M-matrix.

Now assume $u^m \leqslant u$ and prove $u^{m+1} \leqslant u$. By Theorem 2.1 we know that

$$u \leqslant Bu, \quad (A^1u - f^1, v - u) \geqslant 0, \quad \forall v \in Bu.$$

Hence, u is also the solution of the following variational inequality: $u \in K_i$ and

$$(A^1u - f^1, v - u) \geqslant 0, \quad \forall v \in K_i, \tag{5.5}$$

where $K_i = \{v \in \mathbb{R}^n : v_s = u_s \text{ if } s \in N\backslash N_i,\ v_s \leqslant (Bu)_s \text{ if } s \in N_i\}$. Comparing (5.5) with (3.4) and using Lemma 4.2 (noting that $u^m \leqslant u$ and $Bu^m \leqslant Bu$) we obtain $u^{m,i} \leqslant u,\ i = 1, 2$. So $u^{m+1} \leqslant u$. The proof is complete. □

The following convergence theorem can be easily proved.

Theorem 5.3. *Assume condition B and* (2.6) *are satisfied. The* $\{u^m\}$ *monotonically increasing, convergent to the unique solution of* (1.3).

Proof. The argument is similar to that in Theorem 4.2. The only modification is using exact solution u to replace b in (4.7) according to Lemma 5.2.

When we use Algorithm DDM to solve (1.3) under condition B there is a problem: it is very difficult to choose a subsolution u^0 in this case. Hence, we propose another approach to solve (1.3) approximately: we solve the following equation instead of (1.3):

$$F_tu = \max_{1 \leqslant j \leqslant k} \{A_t^i u_t - f^j\} = 0, \tag{5.6}$$

where A_t^j is defined in the proof of Theorem 5.2, and $t \in [0, 1)$ is very closed to 1. □

Of course we need to estimate the error $u - u_t$. The following theorem has given such a estimation.

Theorem 5.4. *Assume condition B and* (2.6) *are satisfied*, *u and u_t are solution of* (1.3) *and* (5.6), *respectively. Then there exists a constant L independent of t such that*

$$\|u - u_t\|_\infty \leqslant L(1 - t). \tag{5.7}$$

Proof. We know that u_t satisfies (5.1). Split $A_t^* = D_t^* - tC_t^*$, where D_t^* is diagonal and C_t^* is nonnegative, $0 \leqslant t < 1$. Let $\tilde{A}_t^* = D_t^* - C_t^*$. Then any row of $\tilde{A}_t^*$ comes from $A^j,\ j = 1, \ldots, k$. Hence,

$$\tilde{A}_t^* u - f_t^* \leqslant 0. \tag{5.8}$$

Eqs. (5.7) and (5.8) implies $A_t^* u_t - \tilde{A}_t^* u \geqslant 0$ and

$$u_t - u \geqslant (A_t^*)^{-1}(\tilde{A}_t^* - A_t^*)u \tag{5.9}$$

since A_t^* $(0 \leqslant t < 1)$ is an M-matrix. □

Similarly, we know that there exists a matrix A whose any row comes from A^j, $j=1,\dots,k$, such that

$$Au - f = 0. \tag{5.10}$$

Split $A = D - C$, where D is diagonal and C is nonnegative. Let $A_t = D - tC$. Then it is easy to see

$$A_t u_t - f \leqslant 0. \tag{5.11}$$

Eqs. (5.10) and (5.11) implies that $Au - A_t u_t \geqslant 0$ and

$$u_t - u \leqslant A_t^{-1}(A - A_t)u. \tag{5.12}$$

It follows from (5.9) and (5.12) that

$$|u_t - u| \leqslant \max\{|(A_t^*|)^{-1}(\tilde{A}_t^* - A_t^*)u, |A_t^{-1}(A_t - A)u|\}.$$

It is obvious $\tilde{A}_t^* - A_t^* = -(1-t)C_t^*$, $A_t - A = (1-t)C$. Hence,

$$|u - u_t| \leqslant (1-t)\max\{|(A_t^*)^{-1}C_t^* u|, |A_t^{-1}Cu|\}$$

and

$$\|u - u_t\|_\infty \leqslant (1-t)\max\{\|(A_t^*)^{-1}\|_\infty \cdot \|C_t^*\|_\infty, \|A_t^{-1}\|_\infty \cdot \|C\|_\infty\}\|u\|_\infty. \tag{5.13}$$

Since $A_t^* \geqslant \tilde{A}_t^*$, $A_t \geqslant A$ $(0 \leqslant t < 1)$ and all of them are M-matrices, we know that

$$0 \leqslant (A_t^*)^{-1} \leqslant (\tilde{A}_t^*)^{-1}, \quad 0 \leqslant A_t^{-1} \leqslant A^{-1}. \tag{5.14}$$

It is easy to see $\{(\tilde{A}_t^*)^{-1} : 0 \leqslant t < 1\}$ and $\{C_t^* : 0 \leqslant t < 1\}$ both are finite sets. Theorefore, (5.13) and (5.14) implies that there is a constant L independent of t satisfies (5.7).

References

[1] R. Belman, Adaptive Control Processes: A Guided Tour, Princeton University Press, New Jersey, 1961.

[2] F. Camilli, M. Falcone, P. Lanucara, A. Seghini, A domain decomposition method for Bellman equations, in: D.E. Keyes, J.C. Xu (Eds.), Proceedings of DDM 7, AMS, Providence, 1994, pp. 477–484.

[3] R.W. Cottle, J.S. Pang, R.E. Stone, The Linear Complementarity Problems, AP, New York, 1992.

[4] M.G. Crandall, H. Ishi, P.L. Lions, User's guide to viscous solution, Bull. AMS 27 (1992) 1–67.

[5] R. Glowinski, J.L. Lions, R. Tremolieres, Numerical Analysis of Variational Inequalities, North-Holland, Amsterdam, 1981.

[6] B. Smith, P. Bjorstad, W. Gropp, Domain Decomposition, Cambridge University Press, Cambridge, MA, 1996.

[7] M. Sun, Domain decomposition algorithms for solving Hamilton–Jacobi–Bellman equations, Numer, Funct. Anal. Optim. 14 (1993) 145–166.

[8] D. Young, Iterative Solution of Large Linear Systems, AP, New York, 1971.

[9] J.P. Zeng, S.Z. Zhou, On monotinic and geometric convergence of Schwarz methods for two-side obstacle problems, SIAM J. Numer. Anal. 35 (1998) 600–616.